Journal of Differential Equations **168**, 102–128 (2000)
doi:10.1006/jdeq.2000.3880, available online at http://www.idealibrary.com on IDEAL®

Convergence to Traveling Waves in Two Model Systems Related to the Dynamics of Liquid–Vapor Phase Changes

Haitao Fan[1]

Department of Mathematics, Georgetown University, Washington, DC 20057
E-mail: fan@math.georgetown.edu

Received June 4, 1999; revised October 12, 1999

DEDICATED TO PROFESSOR JACK HALE ON THE OCCASION OF HIS 70TH BIRTHDAY

Two prototype systems arising from an earlier model for the dynamics of liquid–vapor phase transitions of retrograde fluids in shock tubes are studied. For the first

sion part of the system is stable. This shows that the admissibility of traveling waves for the original model cannot be determined by stability consideration alone. This leads to the second prototype system for the original model in which the effect of initiation of nuclei is included. For the second system, the convergence to traveling waves is proved for some initial value problems. This result shows the initiation term has a sizable effect in the long time on the speed of the phase boundary.
© 2000 Academic Press

1. INTRODUCTION

In this paper, we shall study the convergence to traveling waves as $t \to \infty$ of the solutions of the initial value problems for the following two systems: The first system is

$$\begin{aligned} &u_t + f(u, \lambda)_x = u_{xx}, \\ &\qquad \lambda_t = \lambda_{xx} + \lambda(\lambda - 1). \end{aligned} \tag{1.1a}$$

The function f satisfies

$$f_{uu} > 0, \qquad f_{u\lambda} \leqslant 0, \qquad \text{and} \qquad f_\lambda > 0. \tag{1.1b}$$

The second system is

$$\lambda_t = \lambda_{xx} + c\lambda_x + \lambda(\lambda - 1) - g(x)\,\lambda, \tag{1.2a}$$

[1] Research supported by NSF Grant DMS-9705723.

0022-0396/00 $35.00

where

$$c \geqslant 2, \qquad g \geqslant 0 \qquad \text{and} \qquad g(x) = 0 \qquad \text{for} \quad x > 0. \tag{1.2b}$$

Above two systems are motivated by the study of the dynamics of liquid–vapor phase transitions in shock tube experiments on retrograde fluids, i.e. fluids with large heat capacity. In [Fan1, Fan2], we studied the isothermal system in Lagrange coordinates describing the motion of retrograde fluids in shock tube experiments:

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p(v, \lambda)_x &= \varepsilon u_{xx}, \\ \lambda_t &= \frac{1}{\gamma}(p - p_e)\,\lambda(\lambda - 1) + w + \mu\lambda_{xx} \end{aligned} \tag{1.3}$$

where v is the specific volume, u the velocity and λ the density fraction of vapor in the vapor–liquid mixture. The term $-w$ in $(1.3)_3$, representing the rate of initiation of liquid nuclei in the mixture. The pressure $p(v, \lambda)$ is assumed to satisfy

$$p_v < 0, \qquad p_\lambda > 0, \qquad \text{and} \qquad p_{vv} > 0. \tag{1.4}$$

We obtained in [Fan2] results on the existence of traveling waves and solutions of some Riemann problems of (1.3) in the limit $\varepsilon \to 0+$ and the scaling $\mu = b\varepsilon$, $\gamma = \varepsilon/a$. The term w is very small away from spinodal region. To make traveling waves possible, we omit the term w in our investigation of traveling waves for (1.3). A traveling wave of (1.3) is a solution of (1.3) of the form $(u, v, \lambda)(\frac{x-ct}{\varepsilon})$ and hence is a solution of

$$\begin{aligned} -cv' &= c^2(v - v_-) + p - p_-, \\ -c\lambda' &= a(p - p_e)\,\lambda(\lambda - 1) + b\lambda'', \\ (v, \lambda)(\pm\infty) &= (v_\pm, \lambda_\pm), \qquad (v', \lambda')(\pm\infty) = (0, 0). \end{aligned} \tag{1.5}$$

where $a = \varepsilon/\gamma$, $b = \mu/\varepsilon$ and "$'$" denotes $d/d\xi$ with $\xi = (x - ct)/\varepsilon$. One of the results on the existence of traveling waves in [Fan2] is that for each fixed $(v_-, \lambda_- = 0)$, if the speed c of traveling waves

$$c \geqslant \omega^* := 2\sqrt{ab\,|p_- p_e|} \tag{1.6}$$

and $p(v_\pm, \lambda_\pm) \geqslant p_e$, then there is a solution of the traveling wave equations (1.5), see Fig. 1.1.

If the speed c satisfies

$$c < 2\sqrt{ab\,|p_+ - p_e|}, \tag{1.7}$$

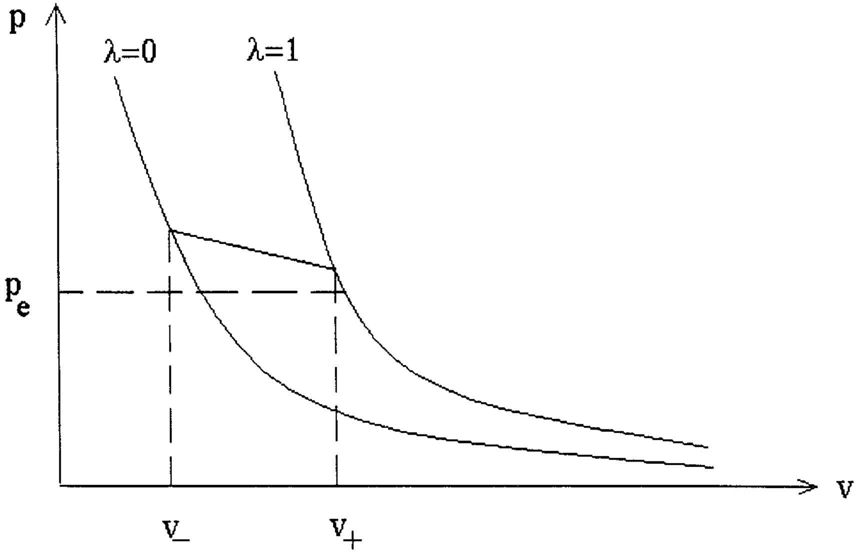

FIGURE 1.1

then there is no traveling wave satisfying $\lambda \in [0, 1]$. Using these traveling waves as the admissible jump discontinuities, we constructed solutions of some Riemann problems in [Fan1, Fan2] which exhibit all major one-dimensional wave patterns observed in [TCK, TCMKS] during actual shock experiments on retrograde fluids. However, there are many questions left unanswered: One such issue is as follows: Each state $(v_-, \lambda_- = 0)$ can be connected to many states $(v_+, \lambda_+ = 1)$ by traveling waves, and thus solutions to Riemann problems are not unique. This nonuniqueness raises the issue of admissibility of traveling waves of (1.3). Only stable waves can exist in a solution for a sizable time. Thus, only stable waves are considered admissible. Therefore, we need to consider the stability of traveling waves in (1.3) to decide the admissibility of the traveling wave.

To further study the stability of traveling waves of (1.3), we notice that system (1.3) is a combination of viscous conservation laws and a reaction diffusion equation. We ask which part is dominant in determining the stability of traveling waves of (1.3). To answer this question, we observe that, when the end states of the traveling wave are as shown in Fig. 1.1, the effect of the factor $p(v, \lambda) - p_e$ is minor if $p - p_e > \delta > 0$. In this case, we can consider $p(v, \lambda) - p_e$ as a constant. This results in the system (1.1) as a prototype for investigating the stability of traveling waves with end states shown in Fig. 1.1. In fact, system similar to (1.1), with the function $f(u, \lambda) = u^2/2 + \lambda u$ and various source terms, is used extensively in [FD] to model combustion and phase transitions.

One of main effort of this paper is to study the existence and stability of traveling waves of (1.1). Let $(u, \lambda) = (\phi, \psi)(x - ct)$ be a traveling wave solution of (1.1). Assume that ψ is stable with respect to the perturbation in λ. Then, we shall see in this paper that (ϕ, ψ) is stable under some conditions. In other words, the stability of $(u, \lambda) = (\phi, \psi)$ is determined by the reaction-diffusion part of $(1.1)_2$. Thus, an review of known results on the stability of traveling waves of

$$\lambda_t = \tfrac{1}{2}\lambda_{xx} + \lambda(\lambda - 1) \tag{1.8}$$

is in order: Equation (1.8) has traveling waves with $\lambda(-\infty) = 0$, $\lambda(\infty) = 1$ iff the speed of the traveling wave $c \geqslant 2$. We recall from [Br] that the stability of the traveling wave with speed c, denoted by λ_c, depends on the decay rate of the perturbation at $x = \infty$. More precisely, the solution of (1.8) converges in form to the traveling wave with speed $c > \sqrt{2}$ iff for some $h > 0$,

$$\lim_{t \to \infty} \frac{1}{t} \ln \left[\int_t^{(1+h)t} (1 - \lambda(0, y))\, dy \right] = -c + \sqrt{c^2 - 2} < \sqrt{2}, \tag{1.9}$$

and for some $\eta > 0$, $M > 0$, $N > 0$,

$$\int_x^{x+N} (1 - \lambda(0, y))\, dy > \eta \qquad \text{for} \quad x \leqslant -M. \tag{1.10}$$

This and our results on the stability of traveling waves of (1.1) suggests that most of the traveling waves of (1.3) connecting a fixed (v_-, λ_-) to various (v_+, λ_+) are admissible from the stability point of view. Thus, the nonuniqueness of the Riemann problems cannot be settled by stability considerations alone. Some important factors maybe missing in our above considerations of the admissibility of traveling waves of (1.3).

Above consideration brings to our attention the term w, which we neglected in our study of the traveling waves to avoid the "cold boundary" difficulty. This leads us to system (1.2): To fix ideas, we consider the following solution of (1.3) of a Riemann problem simulating the wave splitting when vapor is compressed by a shock: The initial data is

$$v = \begin{cases} v_-, & \text{if} \quad x < 0, \\ v_+, & \text{if} \quad x > 0, \end{cases} \tag{1.11}$$

and $\lambda_- = 0$, $\lambda_+ = 1$ with $u_\pm$ to be chosen later. We choose v_- so that there is a traveling wave of (1.3) connecting $(v_-, \lambda_- = 0)$ to $(v^*, \lambda_+ = 1)$, see Fig. 1.2.

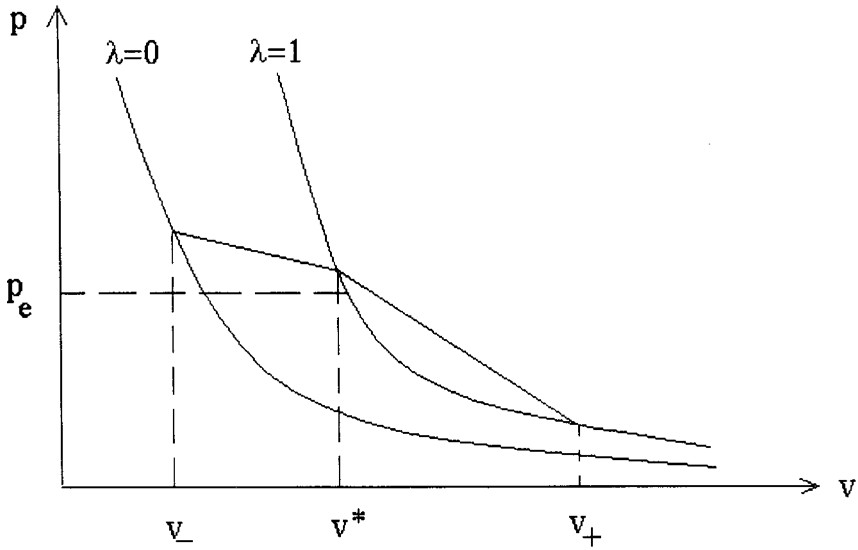

FIGURE 1.2

Then we choose v_+ so that the slope of the chord connecting $(v^*, p(v^*, 1))$ and $(v_+, p(v_+, 1)$ is greater than c. With $u_\pm$ chosen by corresponding Rankine-Hugoniot conditions, the Riemann problem with such data has a solution as depicted in Fig. 1.3 which consists of a fore-runner nonreacting shock, FS, followed by a slower moving condensation

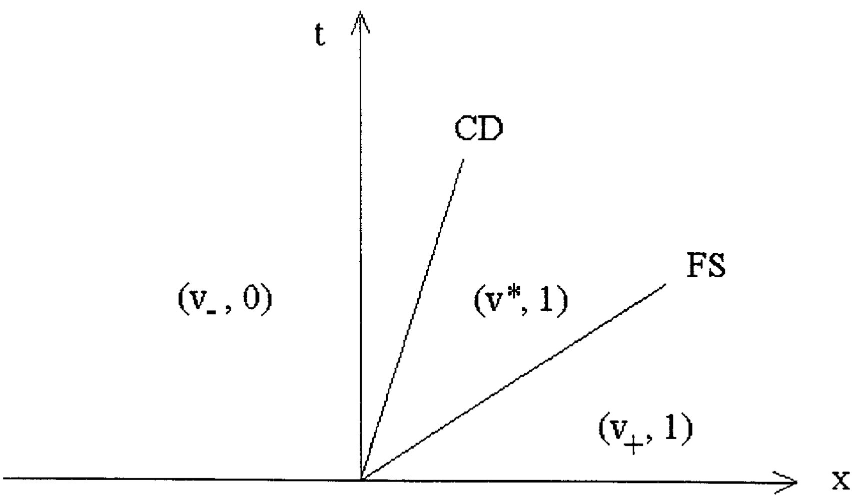

FIGURE 1.3

shock, CD. This is one of the typical wave patterns observed in actual shock tube experiments on retrograde fluids, see [TCK, TCMKS].

The leading shock sends the stable vapor to metastable vapor. Liquid drops then start to form in the back of the leading shock, which perturbed the state $\lambda=1$ into $\lambda<1$. Then the growth term $(p-p_e)\,\lambda(\lambda-1)/\gamma$ in (1.3) starts to be effective. The term $-w$ in (1.3) represents the rate of initiation of liquid drops in the mixture. It is clear that the larger the $-w$ and longer it is effective, the more λ is changed to $\lambda<1$ and hence, by the stability result on the traveling wave of (1.8), the faster the speed of the stable traveling wave. Therefore, we expect w to play a significant role in deciding the speed of the phase boundary. To gain insight into this situation, we consider the equation

$$\lambda_t=\lambda_{xx}+\lambda(\lambda-1)-g(x-ct)\,\lambda.$$

Here $x=ct$ is the location of the forerunner shock. The term $w=-g(x-ct)\,\lambda$, with $g(x)\geqslant 0$ and $g(x)=0$ for $x>0$, represents the rate of initiation of liquid drops occurring after the forerunner shock. Since the forerunner shock is not slower than the phase boundary and the minimum speed of the phase boundary is 2, it is natural to take $c\geqslant 2$. This equation is relevant to (1.3) as long as $p-p_e>\delta>0$ for some constant $\delta>0$. After the change of variable $x-ct\mapsto x$, we obtain (1.2).

In this paper, we shall prove the existence of traveling waves of (1.2), which is necessarily stationary waves. We shall also prove that the solution of equation (1.2) with initial value $\lambda=1$, modeling the initial state of shock compression experiments, converges to the largest stationary wave as $t\to\infty$. Precise statements of our results will be given in Theorems 1.1 and 1.2 and Corollaries 4.4 and 4.5.

From these results on the prototype systems (1.1) and (1.2) and (1.8), we conjecture the following behavior of the original system (1.3) with initial value (1.11) simulating the wave splitting when vapor is compressed by a shock of moderate strength: At first, the solution starts to split into a leading ordinary shock, sending the vapor to metastable region, followed by a condensation discontinuity, denoted as CD. At this time, the CD moves with the slowest speed possible, since there is not much time for the w term to be effective yet. As $t\to\infty$, the speed of CD will increase until it either catch up to that of the leading shock, as suggested by our results on (1.2), or the pressure in front of CD $p_*=p_e$ where (1.2) ceases to be a prototype model of (1.3). This indicates that most of the traveling waves of (1.3) with fixed v_- and various v_+, see Fig. 1.1, can be observed in the solution of (1.3) with initial value (1.11), but at different times.

We note that the increase in speeds of the phase boundary during wave splitting for the compression case was not reported in experimental reports

[TCK, TCMKS]. Perhaps their shock tubes are not long enough and the observation points are too few to observe this phenomenon. Whether this phenomenon can be observed in actual experiments for retrograde fluids remains to be seen.

A review of the related results is in order: Systems similar to (1.1) with various functions f and rate equations are used by several authors to study combustion and phase transitions. For example, system

$$u_t + (u^2/2 + q\lambda)_x = \beta u_{xx} \tag{1.12}$$

with various rate equations for λ is used as a prototype for studying combustion and phase changes in [FD]. Another example is

$$\begin{aligned} &u_t + (u^2/2 - q_0 Z)_x = \beta u_{xx} \\ &\qquad Z_x = K_0 \phi(u)\, Z, \qquad t > 0, \qquad -\infty < x < \infty \end{aligned} \tag{1.13}$$

which was used in [Ma] for modeling high Mach number combustion. Extensive mathematical analysis has been carried out on systems (1.12) and (1.13). However, the rate equation $(1.1)_2$ is different from that of (1.12) and (1.13). Phenomena and issues of (1.1) and hence the analysis of (1.1) cannot be covered by that of (1.12) and (1.13).

For the system (1.2), the most relevant earlier works are those on the KPP equation, also called Fisher equation,

$$u_t = u_{xx} + u(1-u). \tag{1.14}$$

Stability in form for traveling waves of (1.14) was proved in [U] for a wide range of perturbation data. The method used in [U] is studying the evolution of phase diagrams. Later, Bramson, [Br], gave a complete characterization of the convergence to traveling waves of (1.14) via the Feyman–Kac formula. Although we can gain insights into (1.2) from the results and the analysis of (1.14), whether the methods used in [U] and [Br] can be used on (1.2) remains to be seen.

The organization and major results of this paper is as follows: In Section 2, we prove the existence of traveling waves of (1.1), summarized in the following theorem:

THEOREM 1.1. *Assume $c \geqslant 2$, $\lambda_- = 0$ and $\lambda_+ = 1$. Then* (1.1) *has traveling wave solutions if* $(u_\pm, \lambda_\pm)$ *satisfies the Rankine–Hugoniot condition*

$$c \geqslant 2, \qquad \text{and} \qquad \lambda_- = 0, \quad \lambda_+ = 1$$

$$c = \frac{f(u_+, \lambda_+) - f(u_-, \lambda_-)}{u_+ - u_-}$$

and one of the following holds:

(i) $u_- > u_+$ *and* $f_u(u_\pm, \lambda_\pm) - c > 0$.

(ii) $u_- < u_+$ *and* $f_u(u_\pm, \lambda_\pm) - c < 0$.

(iii) $u_- > u_+$, $f_u(u_+, \lambda_+) - c < 0$ *and* $f_u(u_-, \lambda_-) - c > 0$.

There are traveling waves of (1.1) which are not listed in above theorem. These traveling waves are nonreacting, i.e. $\lambda_+ = \lambda_-$ whose existence is already covered by classical shock theory. Our interest is in the reacting traveling waves.

In Section 3, we study the stability of traveling waves of (1.1). We prove that the stability of traveling waves $(u, \lambda) = (\phi, \psi)$ is determined by the stability of the λ part. This shows that we need to consider the effect of initiation of nuclei modeled by the term w in (1.3) to determine the movement of phase boundary. The precise statement of the result is as follows:

THEOREM 1.2. *Let* $f \in C^3$ *and* $(u, \lambda) = (\phi, \psi)$ *be a traveling wave of* (1.1) *provided by case* (i) *or* (ii) *of Theorem* 1.1. *Assume that* ψ *is stable in the sense that*

$$\lim_{t \to \infty} \sup_{x \in \mathbb{R}} |\lambda(x, t) - \psi(x - ct)| = 0 \tag{1.15}$$

for the given perturbations in λ *and*

$$\int_0^t (\|\lambda - \psi\|_{L^1} + \|\lambda - \psi\|_{H^2})\, ds \leqslant \eta < \infty \tag{1.16}$$

and $\|\lambda_x\|_{L^\infty} \leqslant C$ *for some constant* η *and* C *independent of* t. *There exist positive constants* c_1 *and* c_2 *such that if* η *is small and*

$$\left| \int_{\mathbb{R}} (u(x, 0) - \phi(x))\, dx \right| \leqslant c_1, \tag{1.17}$$

$$\int_{\mathbb{R}} (1 + x^2)\, |u(x, 0) - \phi(x)|^2\, dx \leqslant c_2 \tag{1.18}$$

and

$$\int_{\mathbb{R}} |u_x(x, 0) - \phi_x(x)|^2\, dx < \infty, \tag{1.19}$$

then there is a unique global solution of (1.1) *with* $u(x, t) \in C^0([0, \infty); H^1) \cap L^2([0, \infty); H^2)$ *satisfying*

$$\lim_{t \to \infty} \sup_{x \in \mathbb{R}} |u(x, t) - \phi(x - ct)| = 0, \tag{1.20}$$

and

$$\lim_{t \to \infty} \int_{\mathbb{R}} |u(x, t) - \phi(x - ct)|^2 \, dx = 0. \tag{1.21}$$

From our analysis in Section 3, we see that the solution $u(x, t)$ will decompose into the traveling wave and a diffusion wave moving with the asymptotic speed $f_u(u_+, \lambda_+)$ carrying the extra mass.

We also note that the condition (1.15) cannot hold for many type of perturbations in λ. For example, if the initial data $\lambda(x, 0)$ is the Heaviside function, then $\lambda(x, t) \to \psi(x - 2t - O(1) \ln t)$, see [KPP]. The extra shift $O(1) \ln t$ will make the diffusion wave large in the long time and the above stability result cannot apply here. In this case, system (1.1) may not be a prototype of (1.3) since $p(u, \lambda) - p_e$ may become nonpositive or ∞ since u will change a lot. Thus, the condition (1.15) is consistent with our purpose of using (1.1) as a tool for studying the behavior of (1.3).

In Section 4, we prove the existence of a family of traveling waves of (1.2). It is clear that traveling waves of (1.2) must be stationary waves. We also prove the convergence to stationary waves for some special initial value problems of (1.2). For the interesting case $\lambda(x, 0) = 1$, the solution $\lambda(x, t)$ converges pointwise to the largest stationary wave. The convergence to traveling waves for general initial data with suitable decay rate as $x \to \infty$ remains to be further investigated.

2. TRAVELING WAVES OF (1.1)

In this section, we shall prove the existence and nonexistence of traveling waves of (1.1) with phase changes.

Traveling waves of (1.1) are determined by the equations

$$\begin{aligned} u' &= -c(u - u_-) + (f(u, \lambda) - f(u_-, \lambda_-)), \\ -c\lambda' &= \lambda'' + \lambda(\lambda - 1), \end{aligned} \tag{2.1}$$

$$(u, \lambda)(-\infty) = (u_-, 0), \qquad (u, \lambda)(\infty) = (u_+, 1)$$

where "$'$" is $d/d\xi$. The second equation is the traveling wave equation for the KPP equation. It is well known that equation $(2.1)_2$ has a solution if and only if

$$c \geqslant 2, \qquad \text{and} \qquad \lambda_- = 0, \qquad \lambda_+ = 1. \tag{2.2}$$

Thus, we assume (2.2) in Sections 2, and 3.

For (2.1) to have a solution, the Rankine–Hugoniot condition

$$c = \frac{f(u_+, \lambda_+) - f(u_-, \lambda_-)}{u_+ - u_-} \tag{2.3}$$

is necessary. This indicates that the equilibrium points $(u_\pm, \lambda_\pm)$ are points of intersection of the line

$$v = c(u - u_-) + f(u_-, \lambda_-)$$

and curves

$$v = f(u, \lambda_\pm)$$

in the (u, v)-plane, as shown in Fig. 2.1. We see that for each $(u_-, \lambda_- = 0)$, there are up to three other equilibrium points under the condition (1.1b).

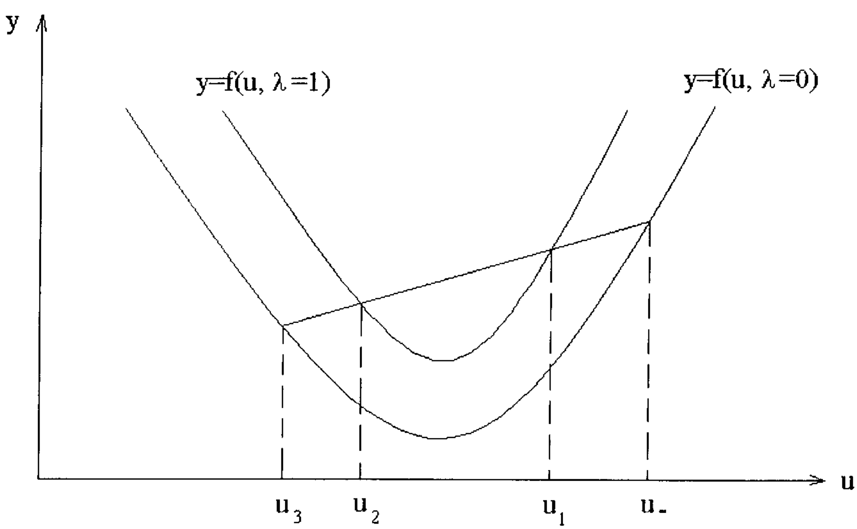

FIGURE 2.1

The eigenvalues at the equilibrium points with $(u, \lambda)=(u, 0)$ are

$$\begin{aligned}\alpha_1(0)&=f_u(u,0)-c,\\ \alpha_2(0)&=\frac{-c+\sqrt{c^2+4}}{2}>0,\\ \alpha_3(0)&=\frac{-c-\sqrt{c^2+4}}{2}<0.\end{aligned}\tag{2.4}$$

The eigenvalues at the equilibrium points with $(u, \lambda)=(u, 1)$ are

$$\begin{aligned}\alpha_1(1)&=f_u(u,1)-c,\\ \alpha_2(1)&=\frac{-c+\sqrt{c^2-4}}{2}<0,\\ \alpha_3(1)&=\frac{-c-\sqrt{c^2-4}}{2}<0.\end{aligned}\tag{2.5}$$

According to conditions (2.2) and (2.3), there are the following four possible cases for $(u_\pm, \lambda_\pm)$ involving changes in λ:

Case 1. $u_- > u_+$ and $f_u(u_\pm, \lambda_\pm)-c>0$. This case is shown in Fig. 2.1 with $u_1 = u_+$.

There is a stable manifold $(u(\xi), \lambda(\xi))$ issued from (u_+, λ_+) in view of (2.4). We claim that near u_+, $u(\xi)>u_+$. Indeed, if otherwise, there would be a point ξ_0 such that $(u, \lambda)(\xi_0)$ is close to (u_+, λ_+), $u(\xi_0)<u_+$ and $u'(\xi_0)>0$. We shall prove this leads to contradiction: Condition (1.1b) implies that

$$f_u(u_-, \lambda_-)\geqslant f_u(u_-, \lambda)\geqslant f_u(u, \lambda)\geqslant f_u(u_+, \lambda)\geqslant f_u(u_+, \lambda_+)$$

for $u_+ \leqslant u \leqslant u_-$ and $\lambda_- = 0 \leqslant \lambda \leqslant 1 = \lambda_+$. In other words, $f_u(u, \lambda)-c>0$ for $u_+ \leqslant u \leqslant u_-$ and $0\leqslant \lambda \leqslant 1$ in this case. By the continuity of f_u, $f_u - c>0$ for u close to u_+ However, this and $(2.1)_1$ leads to a contradiction,

$$0<u'(\xi_0)=(-c+f_u)(u(\xi_0)-u_+)+f_\lambda(\lambda(\xi_0)-\lambda_+)<0,$$

where we used $f_\lambda>0$, $\lambda(\xi)\leqslant 1=\lambda_+$ and $-c+f_u>0$. Above contradiction proves our claim. Thus, there is some sufficiently large ξ_1 so that $u'(\xi_1)<0$ and $u(\xi_1)>u_+$.

We further claim that $u'(\xi) < 0$ for all $\xi < \xi_1$. This is because if otherwise, there would be a point $\xi_2 < \xi_1$ such that $u'(\xi_2) = 0$ and $u''(\xi_2) \leqslant 0$. But from $(2.1)_1$, $f_\lambda > 0$ and $\lambda' > 0$, we have

$$u''(\xi_2) = (-c + f_u)\, u'(\xi_2) + f_\lambda \lambda'(\xi_2) = f_\lambda \lambda'(\xi_2) > 0 \tag{2.6}$$

which is a contradiction. In fact, ξ_1 can be chosen arbitrarily close to $+\infty$ and hence $u'(\xi) < 0$ for all $\xi \in \mathbb{R}$.

Since $u'(\xi) < 0$ for all $\xi \in \mathbb{R}$, as proved in last paragraph, as $\xi \to -\infty$, $u(\xi)$ will either go to ∞ or go to an equilibrium point which is necessarily $(u_-, \lambda_- = 0)$. If $(u, \lambda)(\xi) \to (u_-, \lambda_-)$, it is a desired solution of (2.1). If $u(\xi) \to \infty$ as $\xi \to -\infty$, then there is a point $\xi_3 > -\infty$ such that $u(\xi_3) = u_-$ and $\lambda(\xi_3) > \lambda_- = 0$. Then (2.1) implies

$$u'(\xi_3) = f(u_-, \lambda(\xi_3)) - f(u_-, \lambda_-) > 0$$

which contradicts the proven fact that $u'(\xi) < 0$ for all $\xi < \xi_1$. Thus, there is a solution of (2.1) in this case.

Case 2. $u_- > u_+$, $f_u(u_-, \lambda_-) - c > 0$ and $f_u(u_+, \lambda_+) - c < 0$. See Fig. 2.2

In this case, there is a stable manifold entering (u_+, λ_+) with $u(\xi_0) > u_+$ for some $\xi_0 \in \mathbb{R}$. The rest of the proof for this case is similar to the arguments used in Case 1 starting with the paragraph containing (2.6).

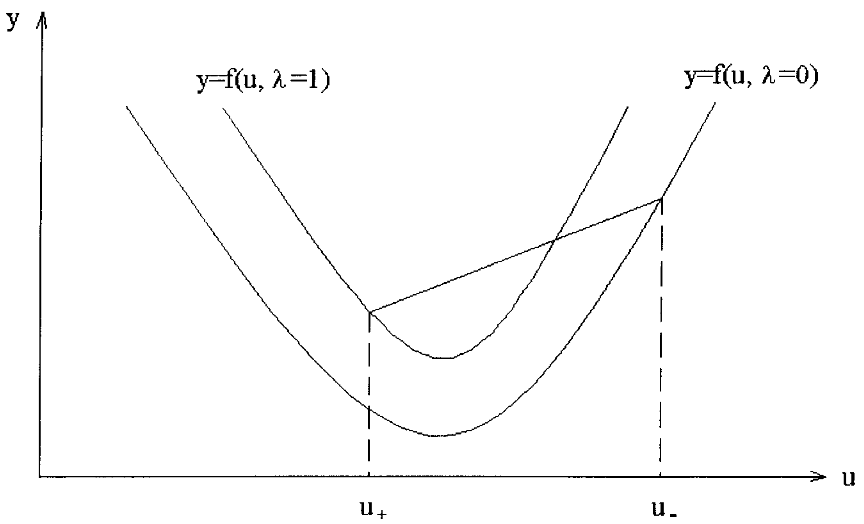

FIGURE 2.2

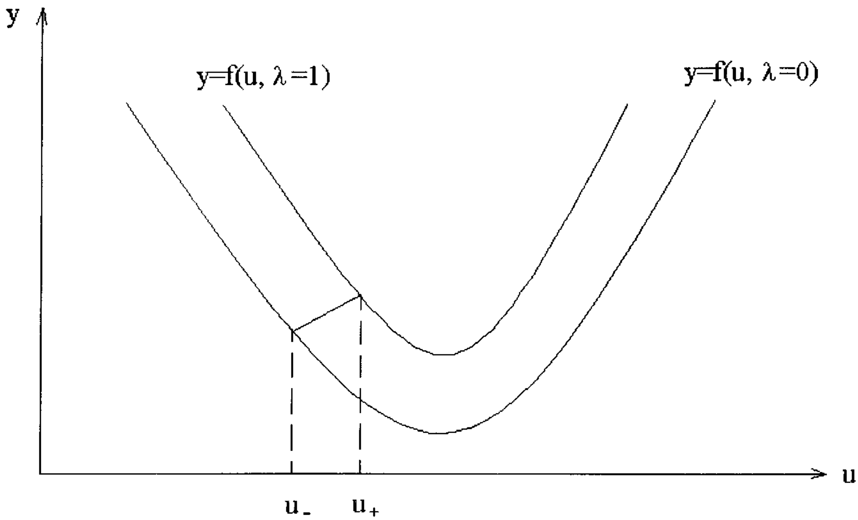

FIGURE 2.3

Case 3. $u_- < u_+$ and $f_u(u_\pm, \lambda_\pm) - c < 0$. See Fig. 2.3.

From (2.4), we know that there is an unstable manifold issued from (u_-, λ_-). Denote this manifold by $(u(\xi), \lambda(\xi))$. Since $f_u(u_-, \lambda_-) - c < 0$, there is a number M such that $f_u(u(\xi), \lambda(\xi)) - c < 0$ for $\xi < M$. By, (2.1), if $u(\xi) < u_-$ for some $\xi < M$, then

$$u'(\xi) = (-c + f_u)(u(\xi) - u_-) + f_\lambda(\lambda(\xi) - \lambda_-) > 0.$$

Thus, $u(\xi) > u_-$ for all $\xi < M$. Similar to the paragraph containing (2.6), we can prove that $u'(\xi) > 0$ for all $\xi \in \mathbb{R}$. As ξ increases, $u(\xi)$ go to ∞ or u_+. If $u(\xi)$ go to ∞ as ξ increase, there is a point $\infty > \xi_3$ such that $u(\xi_3) = u_+$ and $u'(\xi_3) > 0$. However this contradicts (2.1):

$$u'(\xi_3) = -c(u(\xi_3) - u_+) + f(u_+, \lambda(\xi_3)) - f(u_+, \lambda_+) < 0.$$

Thus, as ξ increases, $u(\xi) \to u_+$ and hence (2.1) has a solution.

Case 4. $u_- < u_+$, $f_u(u_-, \lambda_-) < 0$ and $f_u(uok_+, \lambda_+) - c > 0$. See Fig 2.4.
We claim that there is no solution of (2.1) in this case. For the solution of (2.1) to exist, an unstable manifold issued from (u_-, λ_-) has to pass u_1 to reach u_+, where u_1 is the solution of the Rankine–Hugoniot condition

$$-c(u - u_-) + f(u, \lambda_-) - f(u_-, \lambda_+) = 0 \tag{2.7}$$

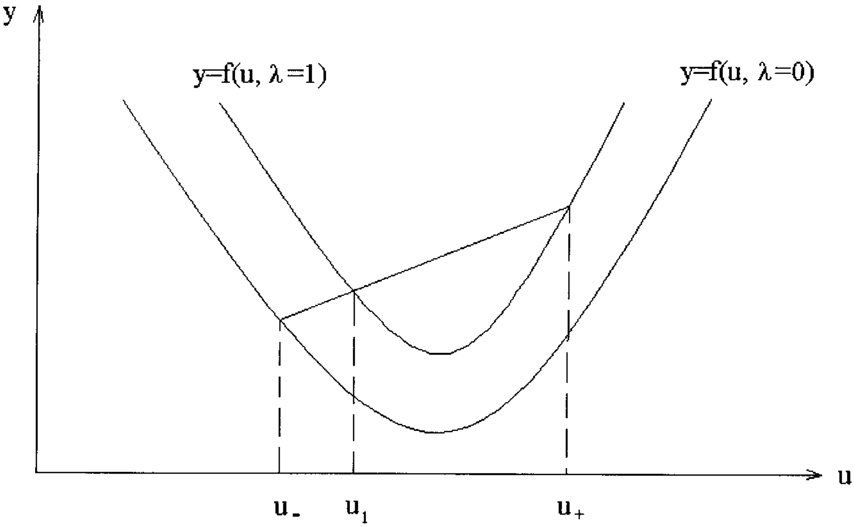

FIGURE 2.4

between $u_{\pm}$, see Fig. 2.4. Thus, there is a point $\xi_3 \in \mathbb{R}$ such that $u(\xi_3) = u_1$. At $\xi = \xi_3$, we have from (2.1) and (2.7) that

$$u'(\xi_3) = -c(u_1 - u_-) + f(u_1, \lambda(\xi_3)) - f(u_-, \lambda_-)$$

$$= f(u_1, \lambda(\xi_3)) - f(u_1, \lambda_+ = 1) = f_\lambda(\lambda(\xi_3) - 1) < 0. \tag{2.8}$$

However, by the same arguments in paragraph containing (2.6), we can prove that (2.8) cannot hold. Therefore there is no solution of (2.1) in this case.

Other cases of $(u_{\pm}, \lambda_{\pm})$ permitted by (2.2) and the Ranking–Hugoniot condition (2.3) do not involve changes in λ and hence are not of our interest.

Summarizing above discussions, we obtain Theorem 1.1.

LEMMA 2.1. *Let* $(\phi, \psi)(\xi)$ *be a solution of* (2.1) *listed in Theorem* 1.1. *Then* (ϕ, ψ) *is monotone and* $(\phi, \psi) = O(1)\exp(-\alpha\xi)$ *as* $\xi \to \infty$ *for some constant* $\alpha > 0$.

Proof. The monotonicity of the traveling waves are proven in our discussion of above four cases. From (2.5), we also have $(\phi, \psi) = O(1)\exp(-\alpha\xi)$ as $\xi \to \infty$ for some constant $\alpha > 0$.

3. THE STABILITY OF TRAVELING WAVES OF (1.1)

In this section, we shall investigate the stability of traveling waves $(u, \lambda) = (\phi, \psi)(x - ct)$ of (1.1). It is known that $\psi' > 0$. The stability of ψ depends solely on the perturbation in λ and equation $(1.1)_2$ which is independent of the rest part of system (1.1). We assume the wave ψ is already stable with respect to perturbation in λ. Furthermore, we assume

$$\int_0^t ds \int_{\mathbb{R}} |\lambda - \psi| \, dx \leqslant \eta_1, \tag{3.0a}$$

$$\int_0^t ds \, \|\lambda - \psi\|_{H^1}^2 \leqslant \eta_2 \tag{3.0b}$$

and

$$\|\lambda(\cdot, t)\|_{C^1} \leqslant C_1 \tag{3.0c}$$

for some constant η_1, η_2 and C independent of t. Then the stability problem of traveling wave of (1.1) becomes

$$\begin{aligned} u_t + f(u, \lambda(x, t))_x &= u_{xx}, \\ u(x, 0) &= \phi(x) + u_0(x). \end{aligned} \tag{3.1}$$

For definiteness, we consider the traveling waves of case (i) of Theorem 1.1. In this case

$$f_u(u_\pm, \lambda_\pm) > c \geqslant 2 \text{ and } \phi' < 0. \tag{3.2}$$

We decompose the solution u as

$$u = \phi + \theta r + v, \tag{3.3}$$

where

$$r = 1/f_{uu}(u_+, \lambda_+) \tag{3.4}$$

and θ represents the diffusion wave determined by the Burgers' equation

$$\theta_t - \theta_{xx} + f_u(u_+, \lambda_+)\,\theta_x + \theta\theta_x = 0, \qquad \int_{\mathbb{R}} \theta(x)\,dx = \theta_0. \tag{3.5}$$

A solution of (3.5) is

$$\theta(x, t) = \frac{1}{\sqrt{t+1}}\, h\left(\frac{x - f_u(u_+, \lambda_+)(t+1)}{2\sqrt{t+1}}\right), \tag{3.6a}$$

where

$$h(y) = \frac{(e^{\theta_0/2} - 1)\, e^{y^2}}{\sqrt{\pi} + (e^{\theta_0/2} - 1)\int_y^\infty \exp(-\xi^2)\,d\xi}. \tag{3.6b}$$

LEMMA 3.1. *Let θ be given in* (3.6) *and $a_0 = f_u(u_+, \lambda_+)$. Then*

(i)

$$\theta = O(1)\,\theta_0 (t+1)^{-1/2} \exp\left[-\frac{(x - a_0(t+1))^2}{4(t+1)}\right];$$

(ii)

$$\theta_x = O(1)\,\theta_0 (t+1)^{-1} \exp\left[-\frac{(x - a_0(t+1))^2}{D(t+1)}\right]$$

where $D > 4$;

(iii)

$$|\theta_t + a_0\theta_x| + |\theta_{xx}| = O(1)\,\theta_0 (t+1)^{-3/2} \exp\left[-\frac{(x - a_0(t+1))^2}{D(t+1)}\right].$$

Proof. Omitted.

Plunging (3.3) into (3.1) and using our choices of θ and r, we obtain

$$v_t + [f_u(\phi + \theta r, \psi)\, v]_x = v_{xx} + R_x, \tag{3.7}$$

where

$$R_x = (\theta_{xx} - \theta_t)\, r - (f_u(\phi + \theta r, \lambda) - f_u(\phi, \psi))_x + [O(1)\, v^2]_x. \tag{3.8}$$

Using the fact $a_0 > c$, we can find a constant $\delta > 0$ such that

$$(|\phi - u_+| + |\psi - \lambda_+|)\,\theta(x, t) = O(1)\,\theta_0 e^{-\delta(t + |x|)}. \tag{3.9}$$

From this and (3.4) and (3.5), we can simplify (3.8) as

$$R = O(1)(\lambda - \psi) + O(1)(v^2 + \theta^3) + O(1)\,\theta_0 e^{-\delta(t + |x|)}. \tag{3.10}$$

The $O(1)$s in (3.8 - 10) depend on ϕ, ψ, θ, v and λ. Thus, we need the assumption (3.0c) and $\|v\|_{L^\infty} < C$ for some constant C here. The a-priori assumption $\|v\|_{L^\infty} < C$ will be justified later. We introduce the anti-derivative of v

$$w = \int_{-\infty}^{x} v(y, t)\,dy.$$

Then the equation (3.7) becomes

$$w_t + f_u(\phi + r\theta, \lambda)\,w_x = w_{xx} + R. \tag{3.11}$$

To make $w \in L^2$, it is necessary that

$$\begin{aligned} 0 = w(\infty, 0) &= \int_{-\infty}^{\infty} v(y, 0)\,dy \\ &= \int_{-\infty}^{\infty} (u(y, 0) - \phi(y) - \theta(y)\,r)\,dy \\ &= \int_{-\infty}^{\infty} (u(y, 0) - \phi(y))\,dy - \theta_0 r \end{aligned}$$

This can be done by choosing

$$\theta_0 = \frac{1}{r} \int_{-\infty}^{\infty} (u(y, 0) - \phi(y))\,dy. \tag{3.12}$$

From (1.17), we have $\theta_0 \ll 1$ by choosing c_1 small.

In this paper, $\|\cdot\|$ denotes the usual L^2 norm.

Lemma 3.2. *There is a constant C_1 and $1 > \mu > 0$ such that if*

$$\|w(\cdot, t)\|_{H^1} < \mu, \qquad \text{and} \qquad \|v\|_{L^\infty} < C_1 \qquad \text{for} \quad 0 < t < T, \tag{3.13}$$

then, under the condition (1.1b), *the following estimate holds for* $0<t<T$

$$\|w(\cdot,t)\|^2+\int_0^t ds\int_{\mathbb{R}}\left[\left|\frac{\partial}{\partial x}f_u(\phi,\psi)\right|w^2(x,s)+w_x^2(x,s)\right]dx$$
$$=\|w(\cdot,0)\|^2+O(1)(\theta_0+\eta_1)\,\mu+O(1)\,\theta_0^3. \tag{3.14}$$

Proof. In the following, we ignore the coefficients when it is convenient. Multiplying Ww on (3.11) and integrating over $-\infty<x<\infty$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}Ww^2-\frac{1}{2}\int_{\mathbb{R}}w^2[W_t+(f_u(\phi+r\theta,\psi)\,W)_x+W_{xx}]\,dx$$
$$+\int_{\mathbb{R}}Ww_x^2\,dx=\int_{\mathbb{R}}WwR\,dx. \tag{3.15}$$

By choosing weight function

$$W=\exp\left(-\frac{1}{2}\int_{-\infty}^x\theta(y,t)\,dy\right), \tag{3.16}$$

we have

$$W_t+(f_u(\phi+r\theta,\psi)\,W)_x+W_{xx}$$
$$=[f_u(\phi,\psi)]_x\,W+\frac{O(1)\,\theta_0^2}{(1+t)^{3/2}}+O(1)\,\theta_0 e^{-\delta(t+|x|)}. \tag{3.17}$$

Using this choice of W and assumption (3.13) in (3.15), we obtain

$$\frac{d}{dt}\int_{\mathbb{R}}Ww^2\,dx+\int_{\mathbb{R}}|[f_u(\phi,\psi)]_x|w^2\,dx+\int_{\mathbb{R}}Ww_x^2\,dx$$
$$=\int_{\mathbb{R}}WwR\,dx+\frac{O(1)\,\theta_0^2\mu^2}{(1+t)^{3/2}}+O(1)\,\theta_0\mu^2e^{-\delta t}$$
$$\leqslant\int_{\mathbb{R}}[O(1)\,w_x^2\,Ww+O(1)\,\theta^3\,Ww+O(1)\,|\lambda-\psi|\,Ww]\,dx$$
$$+\frac{O(1)\,\theta_0^2\mu^2}{(1+t)^{3/2}}+O(1)\,\theta_0\mu e^{-\delta t}, \tag{3.18}$$

where we used

$$[f_u(\phi,\psi)]_x=f_{uu}\phi'+f_{u\lambda}\psi'<0 \tag{3.19}$$

resulting from $\psi'>0$ and $\phi'<0$ from Case 1 of last section and $f_{uu}>0$, $f_{u\lambda}\leqslant 0$ from (1.1b).

The first term on the right hand side of (3.18) is

$$\int_{\mathbb{R}} O(1)\, w_x^2 W w\, dx \leqslant O(1)\, \mu \int_{\mathbb{R}} O(1)\, w_x^2 W\, dx \tag{3.20}$$

which is readily absorbed by the third term of on the left hand side since $\mu > 0$ is small. The second term on the right hand side of (3.18) can be written as

$$\begin{aligned}\left|\int_{\mathbb{R}} O(1)\, \theta^3 W w\, dx\right| &\leqslant O(1) \int_{\mathbb{R}} [\theta^{3.4} W + \theta^{2.6} W w^2]\, dx \\ &\leqslant \frac{O(1)\, \theta_0^3}{(t+1)^{1.2}} + \frac{C\theta_0^2}{(t+1)^{1.3}} \int_{\mathbb{R}} W w^2\, dx,\end{aligned} \tag{3.21}$$

where $C > 0$ is a constant independent of t. Plugging (3.20) and (3.21) into (3.18), we get

$$\begin{aligned}&\frac{d}{dt}\left[\exp\left(-\frac{C\theta_0^2}{(t+1)^{0.3}}\right) \int_{\mathbb{R}} W w^2\, dx\right] \\ &\quad + \exp\left(-\frac{C\theta_0^2}{(t+1)^{0.3}}\right) \int_{\mathbb{R}} \left|\frac{\partial}{\partial x} f_u(\phi, \psi)\right| (w^2 + w_x^2(x, t))\, W\, dx \\ &\leqslant \exp\left(-\frac{C\theta_0^2}{(t+1)^{0.3}}\right) \\ &\quad \times \left(O(1) \int_{\mathbb{R}} |\lambda - \psi|\, W w\, dx + \frac{O(1)(\theta_0^3 + \theta_0^2 \mu^2)}{(1+t)^{1.2}} + O(1)\, \theta_0 \mu e^{-\delta t}\right).\end{aligned} \tag{3.22}$$

Integrating (3.22) with respect to t and using $1 > W > \beta > 0$ for some constant $\beta > 0$, we obtain

$$\begin{aligned}&\|w(\cdot, t)\|^2 + \int_0^t ds \int_{\mathbb{R}} \left[\left|\frac{\partial}{\partial x} f_u(\phi, \psi)\right| w^2(x, s) + w_x^2(x, s)\right] dx \\ &\leqslant \|w(\cdot, 0)\|^2 + O(1)\, \theta_0^3 + O(1)\, \mu(\theta_0 + \eta_1)\end{aligned} \tag{3.23}$$

as desired. ▮

Lemma 3.3. *In addition to the assumptions in Lemma 3.2, if*

$$\|w_x(\cdot, t)\|_{L^\infty} < \mu_1 \qquad \text{for} \quad 0 < t < T \tag{3.24}$$

for some sufficiently small constant $1 > \mu_1 > 0$, *then*

$$\|w_x(\cdot, t)\|^2 + \int_0^t ds \int_{\mathbb{R}} w_{xx}^2(x, s)\, dx$$
$$\leqslant O(1)\, \|w_x(\cdot, 0)\|_{H^1}^2 + O(1)(\theta_0 + \eta_1 + \eta_2). \tag{3.25}$$

Proof. Taking $\partial/\partial x$ on (3.11), multiplying the result by w_x and integrating over $\mathbb{R}$, we obtain

$$\frac{d}{dt} \int_{\mathbb{R}} w_x^2\, dx + \|w_{xx}\|^2 = \int_{\mathbb{R}} R_x w_x\, dx - \int_{\mathbb{R}} [f_u(\phi + \theta r, \psi)]_x w_x^2\, dx. \tag{3.26}$$

The inequality

$$\int_{\mathbb{R}} R_x w_x\, dx = -\int_{\mathbb{R}} R w_{xx}\, dx \leqslant \int_{\mathbb{R}} \left(\frac{1}{\varepsilon} R^2 + \varepsilon w_{xx}^2\right) dx$$

with ε small enough and (3.26) yield

$$\|w_x(\cdot, t)\|^2 + \int_0^t \|w_{xx}(\cdot, s)\|^2\, ds$$
$$\leqslant \|w_x(\cdot, 0)\|^2 + O(1) \int_0^t ds \int_{\mathbb{R}} R^2(x, s)\, dx + O(1) \int_0^t ds \int_{\mathbb{R}} w_x^2\, dx. \tag{3.27}$$

From the definition of R, (3.8), we have

$$R^2 \leqslant O(1)(\lambda - \psi)^2 + O(1)\, \theta^6 + O(1)\, \theta_0 e^{-\delta(t + |x|)} + O(1)\, w_x^4. \tag{3.28}$$

The integration $\int_0^t ds \int_{\mathbb{R}} dx$ of the first three terms of the above yields $O(1)(\theta_0 + \eta_2)$. Applying (3.24) and Lemma 3.2 to the integration of the last term of (3.28) and (3.27) gives $O(1)(\|w(\cdot, 0)\|^2 + \theta_0^3 + \mu\theta_0 + \mu\eta_1)$. ∎

To justify (3.24) and (3.13), we need to estimate $\|w_{xx}\|$.

LEMMA 3.4. *Under the assumptions in Lemmas* 3.2 *and* 3.3, *the solution of* (3.1) *satisfies*

$$\|w_{xx}(\cdot, t)\|^2 + \int_0^t \|w_{xxx}(\cdot, s)\|^2\, ds$$
$$\leqslant O(1)\, \|w(\cdot, 0)\|_{H^2} + O(1)(\theta_0 + \eta_1 + \eta_2). \tag{3.29}$$

Proof. Let $Z := w_{xx}$. Taking $\partial^2/\partial x^2$ on (3.11), we get

$$Z_t = Z_{xx} - [f_u(\phi + r\theta, \psi)\, w_x]_{xx} + R_{xx}.$$

From (3.29), we derive

$$\begin{aligned}
&\|Z(\cdot, t)\|^2 + \int_0^t \|Z_x(\cdot, s)\|^2\, ds \\
&\quad = \|Z(\cdot, 0)\|^2 + \int_0^t ds \int_{\mathbb{R}} [(f_u(\phi + r\theta, \psi)\, w_x)_x\, Z_x + R_x Z_x]\, dx \\
&\quad =: I + II. \qquad (3.30)
\end{aligned}$$

The term I is

$$\begin{aligned}
I &:= \int_0^t ds \int_{\mathbb{R}} (f_u(\phi + r\theta, \psi)\, w_x)_x\, Z_x \\
&\leqslant \frac{1}{\varepsilon} \int_0^t ds \int_{\mathbb{R}} [(f_u(\phi + r\theta, \psi)\, w_x)_x]^2\, dx + \varepsilon \int_0^t ds \int_{\mathbb{R}} Z_x^2\, dx \\
&= \frac{O(1)}{\varepsilon} \int_0^t ds \int_{\mathbb{R}} (w_x^2 + w_{xx}^2)\, dx + \varepsilon \int_0^t ds \int_{\mathbb{R}} Z_x^2\, dx \\
&= O(1)(\|w(0, \cdot)\|_{H^1}^2 + \theta_0 + \eta_1 + \eta_2) + \varepsilon \int_0^t ds \int_{\mathbb{R}} Z_x^2\, dx, \qquad (3.31)
\end{aligned}$$

where in the last step, we used (3.14) and (3.25). The term II in (3.30) is

$$II := \int_0^t ds \int_{\mathbb{R}} R_x Z_x\, dx \leqslant \frac{1}{\varepsilon} \int_0^t ds \int_{\mathbb{R}} R_x^2\, dx + \varepsilon \int_0^t ds \int_{\mathbb{R}} Z_x^2\, dx \qquad (3.32)$$

Recalling the definition of R, (3.8) and (3.10), we see that

$$\begin{aligned}
\int_0^t ds \int_{\mathbb{R}} R_x^2\, dx \leqslant \int_0^t ds \int_{\mathbb{R}} \{[O(1)\, \theta_0 e^{-\delta(t + |x|)} \\
+ O(1)(\lambda - \psi) + O(1)\, \theta^3 + O(1)\, w_x^2]_x\}^2\, dx \qquad (3.33)
\end{aligned}$$

where $O(1)$s depend on $\phi, \psi, \theta, \lambda$ and $v = w_x$. Using (3.24), (3.13) and (3.0c), we have

$$
\begin{aligned}
&\int_0^t ds \int_{\mathbb{R}} R_x^2\, dx \\
&\quad \leqslant \int_0^t ds \int_{\mathbb{R}} \left[O(1)(\theta_0 e^{-\delta(t+|x|)} + |\lambda-\psi|^2 + |\lambda_x - \psi_x|^2 + |\theta_x^2 \theta^4|) \right. \\
&\qquad \left. + (O(1)\, w_x w_{xx} + O(1)\, w_x^2)^2 \right] dx \\
&\quad \leqslant \int_0^t ds \int_{\mathbb{R}} O(1)[\theta_0 e^{-\delta(t+|x|)} + |\lambda-\psi|^2 \\
&\qquad + |\lambda_x - \psi_x|^2 + |\theta_x^2 \theta^4| + \mu^2 (w_x^2 + w_{xx}^2)]\, dx. \qquad (3.34)
\end{aligned}
$$

Applying (3.14) and (3.25) to (3.34), we obtain

$$
\int_0^t ds \int_{\mathbb{R}} R_x^2\, dx \leqslant O(1)(\theta_0 + \eta_1 + \eta_2 + \|w(\cdot, 0)\|_{H^1}^2). \qquad (3.35)
$$

By plugging (3.31), (3.32) with sufficiently small $\varepsilon > 0$ and (3.35) into (3.30), we conclude (3.29). ∎

Lemma 3.5. *If θ_0, η_1, η_2 and $\|w(\cdot, 0)\|_{H^1}$ are sufficiently small, and $\|w_{xx}(\cdot, 0)\|$ is bounded. Then the problem (3.11) has a unique global solution in the space $\{w \in C^0([0, \infty); H^2) : w_x \in L^2([0, \infty); H^2)\}$. Furthermore, this solution satisfies*

$$
\begin{aligned}
&\|w(\cdot, t)\|_{H^1}^2 + \int_0^t \|w_x(\cdot, t)\|_{H^1}^2\, ds + \int_0^t ds \int_{\mathbb{R}} \left| \frac{\partial}{\partial x} f_u(\phi, \psi) \right| (w^2(x, s)\, dx \\
&\quad \leqslant O(1) \|w(\cdot, 0)\|_{H^1}^2 + O(1)(\theta_0 + \eta_1 + \eta_2), \qquad (3.36)
\end{aligned}
$$

$$
\begin{aligned}
&\|w_{xx}(\cdot, t)\|^2 + \int_0^t \|w_{xxx}(\cdot, s)\|^2\, ds \\
&\quad \leqslant O(1) \|w(\cdot, 0)\|_{H^2} + O(1)(\theta_0 + \eta_1 + \eta_2), \qquad (3.37)
\end{aligned}
$$

$$
\|w(\cdot, t)\|_{C^1} < \mu \qquad (3.38)
$$

for some constant $\mu > 0$.

The proof of above lemma is by the standard continuity argument for parabolic systems. We omit the details.

Proof of Theorem 1.2 The condition (1.18) and the Poincare inequality

$$
\left| \int_0^{\pm\infty} \left(\int_x^{\pm\infty} g(y)\, dy \right)^2 dx \right| \leqslant 4 \left| \int_0^{\pm\infty} y^2 g^2(y)\, dy \right|
$$

implies that $\|w(\cdot, 0)\|_{H^1}$ is sufficiently small. The boundedness of $\|w_{xx}(\cdot, 0)\|$ follows from (1.19). The existence part of the statement follows from Lemma 3.5. Similar to the proof of (3.27), we have from (3.26), (3.28) and (3.38) that for $t > t_1 > 0$,

$$\begin{aligned}
&\left| \|w_x(\cdot, t)\|^2 - \|w_x(\cdot, t_1)\|^2 \right| \\
&\quad \leqslant O(1) \int_{t_1}^{t} ds \int_{\mathbb{R}} R^2(x, s)\, dx \\
&\quad \leqslant O(1) \int_{t_1}^{t} ds \int_{\mathbb{R}} [O(1)\, e^{-\delta(t+|x|)} + O(1)\, \theta^6 + O(1)(\lambda - \psi)^2]\, dx \\
&\qquad + O(1)\, \mu^2 \int_{t_1}^{t} ds \int_{\mathbb{R}} w_x^2\, dx.
\end{aligned} \tag{3.39}$$

From (3.22), (3.0) and (3.36), we know that the left hand side of (3.39) goes to 0 uniformly in t as $t_1 \to \infty$. Thus, the limit $\lim_{t \to \infty} \|w_x(\cdot, t)\|^2$ exists and hence, from (3.36),

$$\lim_{t \to \infty} \|v(\cdot, t)\| = \lim_{t \to \infty} \|w_x(\cdot, t)\| = 0. \tag{3.40}$$

Since $\|v_x(\cdot, t)\| = \|w_{xx}(\cdot, t)\|$ is bounded, the Sobolev inequality

$$\|v(\cdot, t)\|_{L^\infty}^2 \leqslant 2\, \|v(\cdot, t)\|\, \|v_x(\cdot, t)\|$$

and (3.40) yield

$$\lim_{t \to \infty} \|v(\cdot, t)\|_{L^\infty} = 0. \tag{3.41}$$

Estimates (1.20) and (1.21) are immediate consequences of Lemma 2.1, (3.40) and (3.41).

4. EXISTENCE AND CONVERGENCE TO TRAVELING WAVES OF (1.2)

In this section, we shall prove the existence of stationary waves of (1.2). We shall also prove that solutions of some initial value problems converge to traveling waves.

LEMMA 4.1. *For any* $\lambda_0 \in [0, 1]$, *there is a number* $\lambda_1 > 0$ *such that the solution of*

$$\lambda'' + c\lambda' + \lambda(\lambda - 1) - g(x)\,\lambda = 0, \qquad x < 0,$$
$$\lambda(0) = \lambda_0, \qquad \lambda'(0) = \lambda_1 \tag{4.1}$$

is a unstable manifold of $(4.1)_1$, *i.e.* $(\lambda, \lambda')(-\infty) = (0, 0)$.

The proof of above lemma is based on the continuous dependence of the solution of (4.1) on the initial data. We provide an outline of the proof without the details in the following. For each fixed $\lambda(0) = \lambda_0$, the point of intersection of the trajectory of (4.1) for $x < 0$ and the boundary of the first quadrant of (λ, λ')-plane,

$$\mathscr{B} := \{\lambda = 0, \lambda' > 0\} \cup \{\lambda' = 0, \lambda \geqslant 0\} \tag{4.2}$$

continuously depends on the data $\lambda'(0) = \lambda_1$. When $\lambda'(0) = \lambda_1 > 0$ is small, the trajectory of (4.1) intersects the curve $\mathscr{B}$ at some point $(\lambda > 0, \lambda' = 0)$. When $\lambda'(0) = \lambda_1$ is sufficiently large, the trajectory intersects the curve $\mathscr{B}$ at $(\lambda = 0, \lambda' > 0)$ part of $\mathscr{B}$. By the continuous dependence of the trajectory of (4.1) on initial data, there is a number λ_1 so that the trajectory $(\lambda, \lambda')(x)$ meets the curve $\mathscr{B}$ at $(\lambda, \lambda') = (0, 0)$. By the uniqueness of initial value problem of $(4.1)_1$, $(\lambda, \lambda')(x) = (0, 0)$ occurs at $x = -\infty$. Thus, the statements of Lemma 4.1 holds.

Traveling waves of (1.2) are solutions of (4.1) with $\lambda(-\infty) = 0$, $\lambda(\infty) = 1$.

LEMMA 4.2. *There are traveling waves of* (1.2) *if* $c \geqslant 2$. *Furthermore, these traveling waves* $\lambda(x)$ *satisfy* $\lambda'(x) > 0$ *and*

$$\lambda(x) = O(1)\, e^{-bx} \qquad as \quad x \to \infty,$$

where $b = -\frac{1}{2}(-c + \sqrt{c^2 - 4}) > 0$.

Proof. Consider the triangle in the (λ, λ')-plane

$$D := \left\{(\lambda, \lambda') \in \mathbb{R}^2 : 0 \leqslant \lambda \leqslant 1,\ 0 \leqslant \lambda' \leqslant -\frac{c}{2}(\lambda - 1)\right\}. \tag{4.3}$$

We claim that for $\lambda_0 > 0$ sufficiently small, the (λ_0, λ_1) provided by Lemma 4.1 is inside the triangle D. To this end, it suffices to prove that when $\lambda_0 > 0$ is small, $\lambda_1 > 0$ is also small. Indeed, for each fixed $\delta > 0$, the trajectory of

(4.1) with $(\lambda, \lambda')(0) = (0, \delta)$ intersects the curve $\mathscr{B}$ at $(\lambda, \lambda') = (0, \delta)$. By the continuous dependence of the point of intersection of the trajectory of (4.1) and the curve $\mathscr{B}$, the trajectory of (4.1) with $(\lambda, \lambda')(0) = (\lambda_0, \delta)$ intersects the curve $\mathscr{B}$ at $(\lambda, \lambda') = (0, \lambda' > 0)$ for sufficiently small λ_0. Hence for such small $\lambda_0 > 0$, the trajectory of (4.1) with $(\lambda, \lambda')(0) = (\lambda_0, \delta)$ cannot be an unstable manifold of (4.1) issued from (0, 0). This proves the claim. Thus, at least for each small $\lambda(0) = \lambda_0$, there is an unstable manifold of (4.1) issued from (4.1) such that $(\lambda, \lambda')(0) \in D$. We claim this unstable manifold of (4.1) can be extended to $x \in \mathbb{R}$. and that this unstable manifold $(\lambda, \lambda')(x)$ satisfies $(\lambda, \lambda')(\infty) = (1, 0)$ and hence is a traveling wave of (1.2). To this end, we only need to show that the solution of (4.1), $(\lambda, \lambda')(x)$, with $(\lambda(0) > 0, \lambda'(0) > 0) \in D$ stays in D for $x > 0$. It is clear that the trajectory $(\lambda, \lambda')(x)$ cannot cross $\lambda = 0$ from inside of D. It cannot cross the part of the boundary of D, $0 \leqslant \lambda \leqslant 1$, $\lambda' = 0$ because on this part of the boundary,

$$\lambda'' = -c\lambda' - \lambda(\lambda - 1) + g(x)\,\lambda = -\lambda(\lambda - 1) + g(x)\,\lambda > 0$$

by (4.1). On the part of the boundary of D, $\{(\lambda, \lambda') \in \mathbb{R}^2 : 0 \leqslant \lambda \leqslant 1, \lambda' = -c(\lambda - 1)/2\}$, the difference of the slope of the trajectory and that of the boundary is

$$\begin{aligned} \frac{d\lambda'}{d\lambda} + \frac{c}{2} &= -c - \frac{\lambda(\lambda - 1)}{\lambda'} + \frac{c}{2} \\ &= -c - \frac{\lambda(\lambda - 1)}{-c(\lambda - 1)/2} + \frac{c}{2} \\ &= -\frac{2}{c}\,(c^2/4 - \lambda) \leqslant 0 \end{aligned} \tag{4.4}$$

with "=" holds only if $(\lambda, \lambda') = (1, 0)$, where we used $c \geqslant 2$ and that $g(x) = 0$ when $x > 0$. It is clear from (4.4) that the trajectory $(\lambda, \lambda')(x)$ cannot cross the boundary of D, $\{(\lambda, \lambda') \in \mathbb{R}^2 : 0 \leqslant \lambda \leqslant 1,\ \lambda' = -c(\lambda - 1)/2\}$, from inside. This proves the existence of traveling waves $w(x)$ of (1.2). The statement $\lambda'(x) > 0$ is an immediate consequence of that the trajectory is in D.

The eigenvalues of (4.1) at $\lambda = 1$ are $\alpha_\pm = (-c \pm \sqrt{c^2 - 4})/2$. A simple calculation shows that the slope of the traveling waves entering into (1, 0) in the (λ, λ')-plane is

$$\frac{d\lambda'}{d\lambda} = \alpha_\pm. \tag{4.5}$$

From the last paragraph, we know that the traveling waves we found there stays in the triangle D. Then the slope $\frac{d\lambda'}{d\lambda}$ cannot be less than $-c/2$ and hence

$$\frac{d\lambda'}{d\lambda} = \alpha_+ = -b < 0. \tag{4.6}$$

This proves that $w(x) = O(1)\, e^{-bx}$ as $x \to \infty$. ∎

Note that $(4.1)_1$ is not invariant under shifting. Traveling waves $\lambda(x)$ of (1.2) with different $\lambda(0)$ are different in general.

LEMMA 4.3. *If the initial value $\lambda_0(x)$ of* (1.2) *satisfies*

$$\lambda_{0xx} + c\lambda_{0x} + \lambda_0(\lambda_0 - 1) - g(x)\, \lambda_0 \leqslant 0, \qquad (\geqslant 0), \tag{4.7}$$

then the solution $\lambda(x, t)$ of (1.2) *with this initial value is decreasing* (*increasing*) *with respect to t for each fixed x.*

Proof. By (1.2), the function $v = \lambda_t$ is the solution of

$$\begin{aligned} v_t &= v_{xx} + cv_x + (2\lambda - 1)\, v - g(x)\, v \\ v(x, 0) &= \lambda_t(x, 0) = \lambda_{0xx} + c\lambda_{0x} + \lambda_0(\lambda_0 - 1) - g(x)\, \lambda_0. \end{aligned} \tag{4.8}$$

By the classical maximum principle type of argument, $v \leqslant 0$ (or $v \geqslant 0$) if its initial value is $\leqslant 0$ (or $\geqslant 0$). Then condition (4.7) implies that $\lambda_t \leqslant 0$ (or $\geqslant 0$) for all $(x, t) \in \mathbb{R} \times \mathbb{R}^+$. ∎

Since our original motivation for studying (1.2) is the shock induced phase transitions in pure vapor, the case $\lambda_0(x) = 1$ is particularly interesting.

COROLLARY 4.4. *Let $\lambda(x, t)$ be the solution of* (1.2) *with initial value $\lambda(x, 0) = 1$. This solution converges to a traveling wave $w(x)$ of* (1.2) *as $t \to \infty$. Moreover, this traveling wave is the largest in the sense that all other traveling waves of* (1.2) $\leqslant w(x)$.

Proof. It is clear that when $\lambda_0(x) = \lambda(x, 0) = 1$,

$$\lambda_{0xx} + c\lambda_{0x} + \lambda_0(\lambda_0 - 1) - g(x)\, \lambda_0 = -g(x) \leqslant 0.$$

From Lemma 4.3, $\lambda(x, t)$ is decreasing as t increases. Let $w_1(x)$ be any traveling wave of (1.2). It is clear that $w_1(x) \leqslant 1 = \lambda_0(x)$. By comparison theorem, $\lambda(x, t) \geqslant w_1(x)$ for all $t > 0$. Thus, $\lambda(x, t)$ converges to a function $w(x) \geqslant w_1(x)$. This function $w(x)$ is necessarily a traveling wave of (1.2) or constant 1 or 0. The presence of $-g(x)\, \lambda$ in (1.2) rules out the constant 1 and $w(x) \geqslant w_1(x)$ make $w = 0$ impossible. Thus, $w(x)$ is a traveling wave of (1.2). It is clear from above that $w \geqslant w_1$ for any traveling waves of (1.2) and hence w is the largest traveling wave of (1.2). ∎

COROLLARY 4.5. *Assume $g' \leqslant 0$. Let $\lambda(x, t)$ be the solution of* (1.2) *with initial value $w(x - x_0)$ where $w(x)$ is an increasing traveling wave of* (1.2). *Then this solution converges to a traveling wave of* (1.2) *as $t \to \infty$.*

Proof. The initial value $\lambda_0(x) = w(x - x_0)$ satisfies

$$\lambda_{0xx} + c\lambda_{0x} + \lambda_0(\lambda_0 - 1) - g(x - x_0)\,\lambda_0 = 0.$$

Then we have

$$\begin{aligned}\lambda_{0xx} + c\lambda_{0x} + \lambda_0(\lambda_0 - 1) - g(x)\,\lambda_0 \\ = (g(x - x_0) - g(x))\,\lambda_0 = -g'(\theta)\,x_0\lambda_0.\end{aligned} \tag{4.9}$$

By Lemma 4.3 and $g' \leqslant 0$, $\lambda_t \geqslant 0$ if $x_0 > 0$ and $w(x)$ is a upper bound of $\lambda(x, t)$. If $x_0 < 0$, then $\lambda_t \leqslant 0$ and $w(x)$ is a lower bound of $\lambda(x, t)$. In either case, $\lambda(x, t)$ converges to a traveling wave of (1.2) as $t \to \infty$.

REFERENCES

[Br] M. Bramson, Convergence of solutions of Kolmogorov equation to traveling waves, *Mem. Amer. Math. Soc.* **44** (1983), 285.

[Fan1] H. Fan, Traveling waves, Riemann problems and computations of a model of the dynamics of liquid/vapor phase transitions, *J. Differential Equations* **150** (1998), 385–437.

[Fan2] H. Fan, On a model of the dynamics of liquid/vapor phase transitions, *SIAM J. Appl. Math.*, to appear.

[FD] W. Fickett and W. C. Davis, "Detonation," University of California Press, Berkeley, 1979.

[KPP] A. Kolmogorov, I. Petrovsky, and N. Piskunov, Etude de l'équation de la diffusion avec croissance de la quantité de la matière et son application à un problème biologique, *Moscow Univ. Bull. Math.* **1** (1937), 1–25.

[Ma] A. Majda, A quantitative model for dynamic combustion, *SIAM J. Appl. Math.* **41** (1981), 70–93.

[TCK] P. A. Thompson, G. C. Carofano, and Y.-G. Kim, Shock waves and phase changes in a large-heat-capacity fluid emerging from a tube, *J. Fluid Mechanics* **166** (1986), 57–92.

[TCMKS] P. A. Thompson, H. Chaves, G. E. A. Meier, Y.-G. Kim, and H.-D. Speckmann, Wave splitting in a fluid of large heat capacity, *J. Fluid Mechanics* **185** (1987), 385–414.

[U] K. Uchiyama, The behavior of solutions of some non-linear diffusion equations for large time, *J. Math. Kyoto Univ.* **18** (1978), 453–508.